REPORT SAMSO-TR-75-289

(12)

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Observations on Approximate and Exact Treatments of Fracture Statistics for Polyaxial Stress States

ADA019572

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4 December 1975

Final Report

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Prepared for

OFFICE OF NAVAL RESEARCH
Washington, D. C. 20390

and

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Arlington, Va. 22209

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: SAMSO-TR-75-289
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): OBSERVATIONS ON APPROXIMATE AND EXACT TREATMENTS OF FRACTURE STATISTICS FOR POLYAXIAL STRESS STATES
5. TYPE OF REPORT & PERIOD COVERED: Final
6. PERFORMING ORG. REPORT NUMBER: TR-0076(9990)-2
7. AUTHOR(s): Samuel B. Batdorf
8. CONTRACT OR GRANT NUMBER(s): F04701-75-C-0076
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Aerospace Corporation, El Segundo, Calif. 90245
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 17p.
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Washington, D.C. 20390
12. REPORT DATE: 4 December 1975
13. NUMBER OF PAGES: 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Space and Missile Systems Organization, Air Force Systems Command, Los Angeles, Calif. 90009
15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:

16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):

Fracture Statistics
Failure Statistics
Failure Theory
Brittle Fracture

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An approximate technique sometimes used for estimating the probability of failure of brittle materials under polyaxial loading conditions is to assume that the effect of each principal stress is independent of the presence of the others. It is shown that this assumption is generally unconservative, and that the errors can be large. An exact treatment is given in terms of Weibull theory for materials that obey Weibull's two-parameter formulation, and a simple conservative approximation is suggested for such materials.

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OBSERVATIONS ON APPROXIMATE AND EXACT TREATMENTS OF FRACTURE STATISTICS FOR POLYAXIAL STRESS STATES

In his original treatment of the statistics of fracture, Weibull [1] introduced a two-parameter functional form for the relationship between simple tension and probability of fracture. He also showed how to compute the corresponding statistics of failure for bending, torsion, and other stress states that involve only uniaxial tensile stresses. These applications have become well known and are widely used [2-4].

In problems that involve biaxial or triaxial tensile stresses, the situation is more complicated. Weibull gave, without formal proof, a procedure for treating such problems and showed how to apply it in several simple cases. Some investigators have expressed doubts concerning the rigor of Weibull's treatment of polyaxial stress states [5, 6], and there are indications that these doubts were later shared by Weibull himself [7, 8]. Since, in addition, rather tedious calculations are required for each polyaxial stress state, in practical structures that involve continuously varying stress states, there is a natural tendency to use approximations. One simple approximation, which constitutes, in fact, the only technique suggested for handling polyaxial stress statistics in a recent treatise on fracture [9], is to assume that

$$P_s(\sigma_1, \sigma_2, \sigma_3) = P_s(\sigma_1) P_s(\sigma_2) P_s(\sigma_3) \tag{1}$$

where P_s is the probability of survival, and σ_1, σ_2, σ_3 are the principal stresses. This method was mentioned earlier by Barnett et al. [5]. For brevity, it is referred to here as the Barnett-Freudenthal (BF) approximation.

One objective of the present investigation is to investigate the limitations of this approach and thus assist potential users in understanding the general nature and approximate magnitude of the errors involved. Another objective

is to point out an alternative approximation that is preferable in some applications. For the sake of simplicity and brevity, interest is focused primarily on biaxial tension applied to isotropic materials whose fracture statistics in simple tension can be described with satisfactory accuracy by Weibull's two-parameter formulation:

$$P_f = 1 - \exp[-k\sigma^m] \tag{2}$$

or

$$\ln P_s = \ln(1 - P_f) = -k\sigma^m \tag{3}$$

where P_f is the probability of fracture. An exact treatment of the Weibull theory solution is also given.

The BF approximation would be strictly correct if all the cracks were oriented with their planes normal to any one of the principal stresses. However, this is rarely the case and certainly does not occur in isotropic materials. Generally, there will be some cracks inclined to the principal axes that will be fractured by combined stresses even though capable of surviving any of the principal stresses applied individually. This phenomenon tends to make eq. (1) unconservative. However, there are also cases in which a given crack would be fractured by more than one of the principal stresses even acting alone, and this tends to make the assumption of noninteraction, i.e., eq. (1), conservative. We show how these opposite trends interact, and that, generally, the unconservative tendency dominates.

In the case of equibiaxial tension, eq. (1) reduces to

$$P_s(\sigma, \sigma) = [P_s(\sigma, o)]^2 \tag{4}$$

or

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = 2 \quad . \tag{5}$$

Equation (5) implies that, for the small values of P_f of primary interest

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = \frac{\ln[1 - P_f(\sigma, \sigma)]}{\ln[1 - P_f(\sigma, o)]} \cong \frac{P_f(\sigma, \sigma)}{P_f(\sigma, o)} = 2 \tag{6}$$

i.e., the probability of failure in equibiaxial tension is twice that for uniaxial tension.

As noted earlier, one would expect, from general principles, that this would sometimes be an underestimate of the probability of equibiaxial fracture. In order to determine when and how much, we must turn to theory. For this purpose, we select a recent reformation of weakest link theory in which the flaws are identified as cracks [10], and the polyaxial stress statistics are derived in a straightforward manner from the basic assumptions. It was noted earlier [8] that, according to this theory, when the probability of fracture in simple tension obeys eq. (2), the probability for equibiaxial tension becomes

$$P_f(\sigma, \sigma) = 1 - \exp\left[-(2m+1)mk\sigma^m \frac{\Gamma(m)\,\Gamma(1.5)}{\Gamma(m+1.5)}\right] \quad . \tag{7}$$

For integral values of m, eq. (7) becomes

$$P_f(\sigma, \sigma) = 1 - \exp\left[-\frac{k\,(m)\,!\,\sigma^m}{(m-0.5)\,(m-1.5)\,\text{--}\,0.5}\right] \quad . \tag{8}$$

Combining eqs. (2) and (8), we obtain

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = \frac{m!}{(m - 0.5)(m - 1.5) --- 0.5} \tag{9}$$

In Fig. 1, this result is compared with the previous one given in eq. (5), and we conclude that the BF approximation leads to the correct result for equibiaxial tension when $m = 1$. However, for $m = 5$ and 10, it underestimates small equibiaxial failure probabilities by factors of approximately 2 and 3, respectively. Note that eq. (9), can be approximated surprisingly well by the very simple equation

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} \cong 2\,(m)^{0.45} \quad . \tag{10}$$

From Fig. 1, it is clear that the BF approximation is conservative for $m < 1$. However, values of m in the range of $0 < m \leq 1$ are abnormal. With the use of eq. (2), it is readily shown that the slope of the $P_f(\sigma)$ curve is zero at $\sigma = 0$ when $m > 1$, in agreement with observation. For $m = 1$, the slope is finite; for $m < 1$, it is infinite; such behavior is rarely, if ever, encountered in nature. Thus, we conclude that, for practical purposes, eq. (1) can be considered unconservative.

We now consider the way the size of the error in eq. (1) varies with stress ratio, again focusing attention on the biaxial case. It was pointed out earlier [8] that, for materials that obey eq. (2), the Weibull theory and that of Batdorf and Crose [10] give the same results. In the general case of triaxial tension characterized by principal stresses σ_1, σ_2 and σ_3, where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq o$, both theories imply that the probability of fracture takes the form

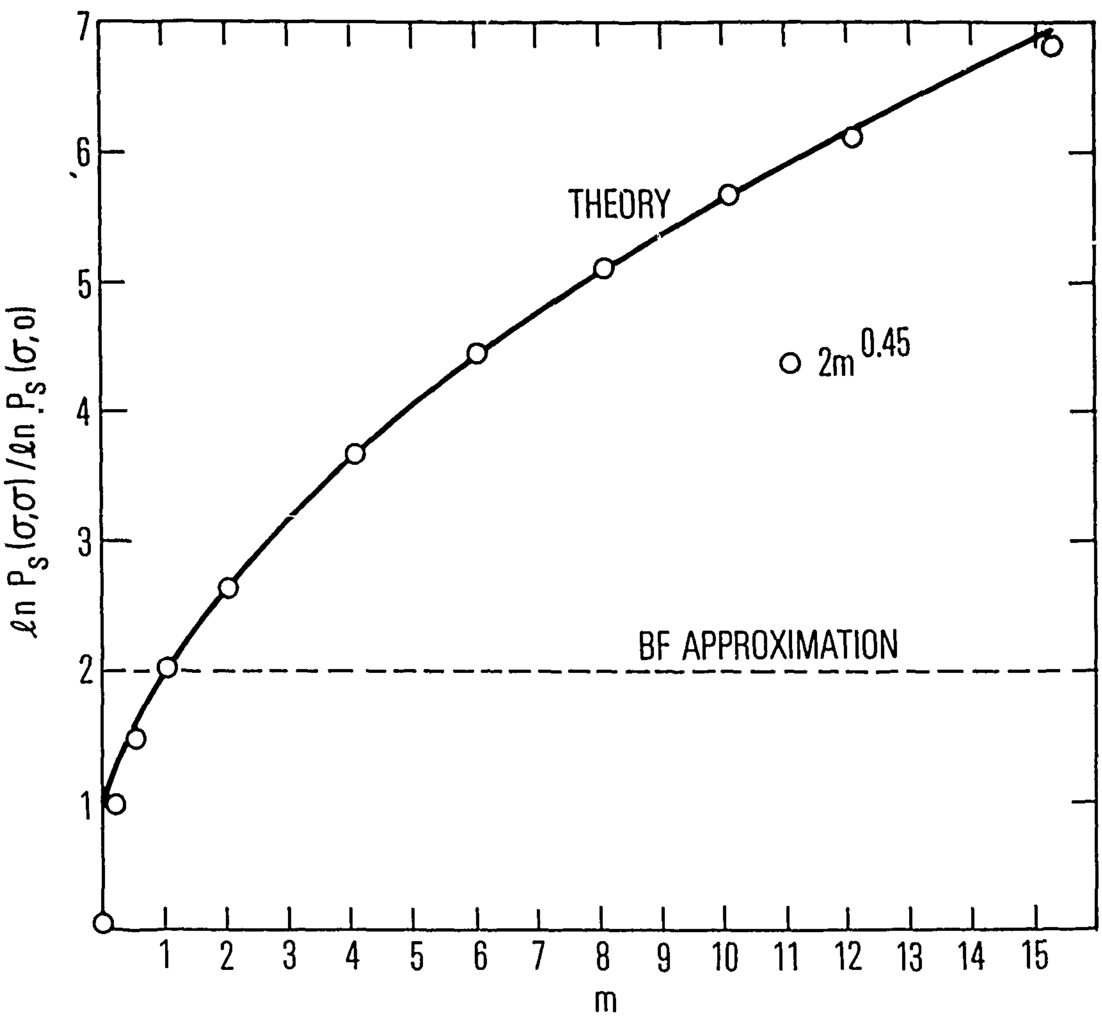

Figure 1. Ratio of ℓn of Probability of Survival in Equibiaxial Tension to that in Uniaxial Tension [equal to $P_f(\sigma, \sigma)/P_f(\sigma, o)$ when $P_f \ll 1$]

$$P_f = 1 - \exp\ -k\int_o^{2\pi} d\phi \int_o^{\pi} (\sigma_1 \cos^2\phi \sin^2\theta + \sigma_2 \sin^2\phi \sin^2\theta$$

$$+ \sigma_3 \cos^2\theta)^m \sin\theta\, d\theta \tag{11}$$

where θ and ϕ are the polar and azimuthal angles, respectively. As noted by Shur [11], the integral in eq. (11) yields

$$\int_o^{2\pi} d\phi \int_o^{\pi} (\ )^m \sin\theta\, d\theta = 4\pi\, \sigma_1^m \frac{(m!)^2}{(2m+1)!} \sum_{i,j,k}^{m} \frac{(2i)!\ (2j)!\ (2k)!}{(i!\ j!\ k!)^2}\, s^j\, t^k \tag{12}$$

where i, j, k are any positive integers that satisfy the inequali'ies $o < i, j, k < m$ and $i + j + k = m$, and where

$$s = \sigma_2/\sigma_1 \tag{13a}$$

$$t = \sigma_3/\sigma_1 \quad . \tag{13b}$$

For the biaxial case, $\sigma_3 = t = o$, and the probability of survival becomes

$$P_s = \exp\left[- \frac{4\pi k}{2m+1} \sigma_1^m F(m, s)\right] \tag{14}$$

where

$$F(m, s) \quad \frac{(m!)^2}{(2m)!} \sum_{i=o}^{m} \frac{(2i)!\,[2(m-i)]!}{[i!(m-i)!]^2} \quad . \tag{15}$$

Thus, in the general biaxial case,

$$\frac{\ln P_s(\sigma_1, \sigma_2)}{\ln P_s(\sigma_1, o)} = \frac{F(m, s)}{F(m, o)} = F(m, s) \tag{16}$$

where m is any positive integer, and $o \leq s = \frac{\sigma_2}{\sigma_1} \leq 1$.

The BF approximation gives, for the biaxial case,

$$\frac{\ln P_s(\sigma_1, \sigma_2)}{\ln P_s(\sigma_1, o)} = \frac{\ln P_s(\sigma_1, s)}{\ln P_s(\sigma_1, o)} = 1 + s^m \quad . \tag{17}$$

Curves that represent eqs. (16) and (17) are shown in Fig. 2. It is evident that, for values of m of practical interest, the BF approximation consistently underestimates the fracture contribution of the second stress, and that the error is largest, in absolute terms, at least, for the equibiaxial case, s = 1.

Another way to compare theory with the BF approximation is to take the ratios of eqs. (16) and (17). Thus,

$$\frac{\ln (P_s)_{Th}}{\ln (P_s)_{BF}} = \frac{F(m, s)}{1 + s^m} \quad . \tag{18}$$

This result is shown in Fig. 3. It is evident that the ratio increases monotonically from s = o to s= 1. Moreover, in the range $2 < m \lesssim 10$, the ratio is reasonably constant near s = 1.

The first of these observations suggests a simple method for obtaining an upper bound to the failure probability, or a lower bound to the survival

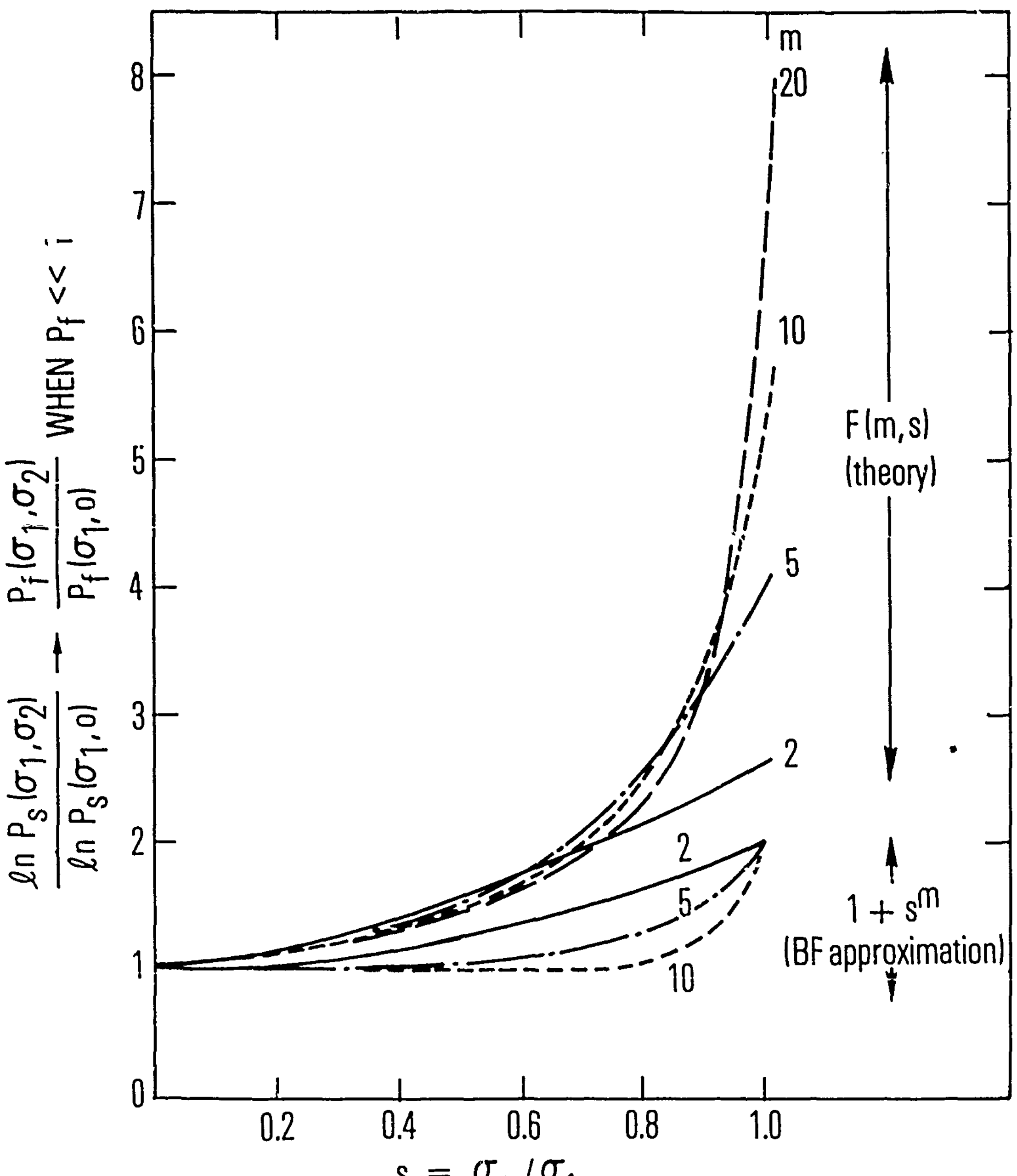

Figure 2. Comparison of Theoretical Survival Prediction with Barn.tt-Freudenthal (BF) Approximation

probability of a biaxially stressed structure; viz., it implies that, in every element of the structure,

$$\frac{\ell n\,(P_s)_{Th}}{\ell n\,(P_s)_{BF}} = \frac{F\,(m,\ s)}{1 + s^m} \leq \frac{F\,(m,\ 1)}{2} \cong m^{0.45} \quad . \tag{19}$$

Since eq. (19) holds for every element individually, it must hold for the structure as a whole.

Thus, we conclude that

$$- \ell n\,(P_s)_{BF} \leq - \ell n\,(P_s)_{Th} \leq - m^{0.45}\,\ell n\,(P_s)_{BF} \quad . \tag{20}$$

For small values of P_f, this implies that

$$(P_f)_{BF} \leq (P_f)_{Th} < m^{0.45}\,(P_f)_{BF} \quad . \tag{21}$$

Because the curves of Fig. 3 are fairly flat near s = 1, in the case of structures whose loading is nearly equibiaxial, the second inequality in eqs. (20) and (21) approaches equality. In the case of circular disks with uniform lateral or concentric ring loading, for instance, the (small) probabilities of failure should, according to the theories discussed here, be quite close to $m^{0.45}$ times the result given by the BF approximation, which yields a simple closed-form solution.

As mentioned earlier, the preceding analysis is based on the assumption that the uniaxial tensile fracture statistics of the material under consideration can be described with adequate accuracy by Weibull's two-parameter equation. When the material is better represented by his three-parameter equation, the analysis becomes more complicated and is beyond the scope of the present paper. However, the general observation that eq. (1) is unconservative still applies, and it is to be expected

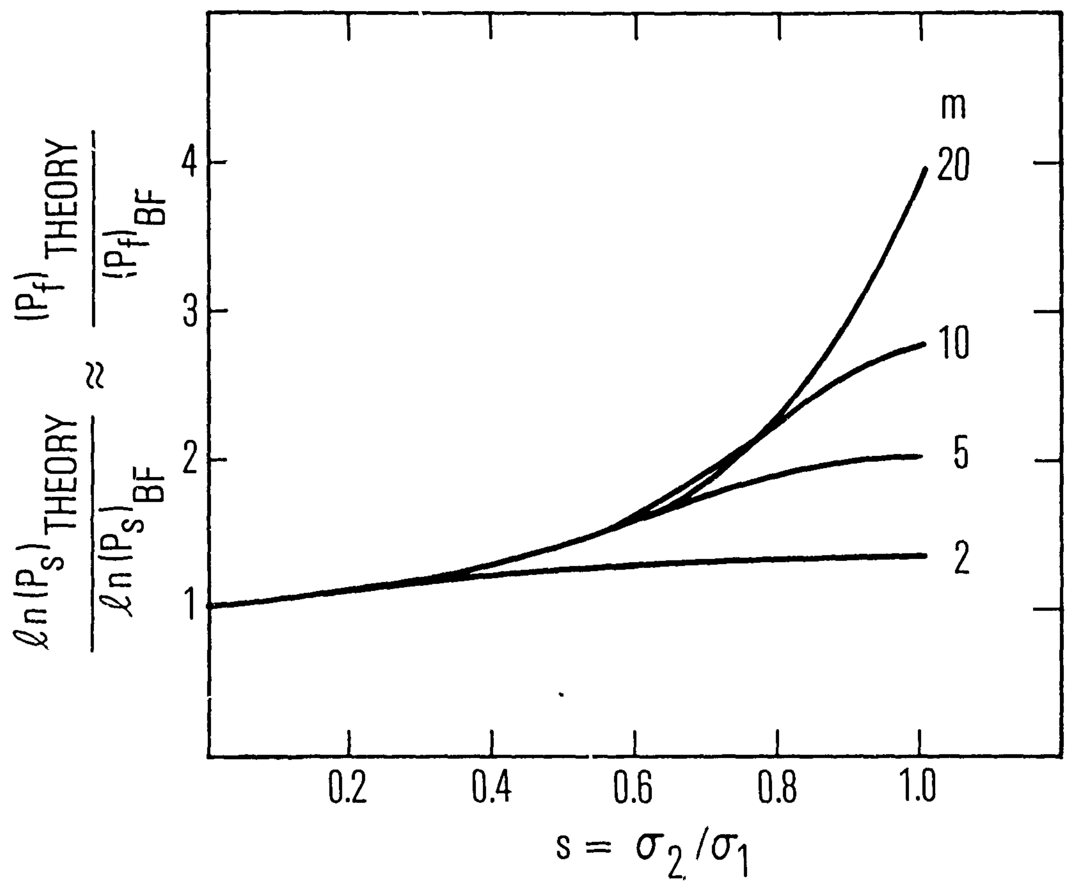

Figure 3. Ratio of Theoretical Prediction to Barnett-Freudenthal (BF) Approximation

that the size of the error might be substantial, particularly when the Weibull parameter m is large. Finally, it should be pointed out that, for materials that obey eq. (2), an accurate Weibull-type treatment of structural problems that involve biaxial or triaxial stresses is not difficult when a computing machine is used in conjunction with finite elements provided that closed-form solutions to the integral in eq. (11) are used. These solutions are given in eqs. (12), (14), and (15).

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